

# A GEOMETRIC FRAMEWORK FOR THE INCONSISTENCY IN PAIRWISE COMPARISONS

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**ABSTRACT.** In this study, a pairwise comparison matrix is generalized to the case when coefficients create Lie group  $G$ , non necessarily abelian. A necessary and sufficient criterion for pairwise comparisons matrices to be consistent is provided. Basic criteria for finding a nearest consistent pairwise comparisons matrix (extended to the class of group  $G$ ) are proposed. A geometric interpretation of pairwise comparisons matrices in terms of connections to a simplex is given. Approximate reasoning is more effective when inconsistency in data is reduced.

**Keywords:** approximate reasoning, inconsistency, Lie group, pairwise comparisons, measure, differential geometry, holonomy, matrix, simplex.

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## 1. INTRODUCTION

The first application of pairwise comparisons is attributed to Llull in the 13th century (see [5]). Deciding which of two objects may fit for purpose is a very natural question. Decision making is a real problem for numerous sciences. Mathematical investigation of pairwise comparisons is of considerable importance since this method has been used in projects of national importance. A mathematical framework should help other researchers to provide answers to problems which surfaced in the past and still remain unsolved (such as the best approximation or inconsistency reduction). Inconsistency in pairwise comparisons is demonstrated by a disagreement of three evaluations in a cycle. Evidently, the ratio of comparing entity  $A$  to  $B$  and  $B$  to  $C$  gives us the ratio of  $A$  to  $C$  as the product of the former ratios. However, collecting three independent assessments of the above ratios may lead to inconsistency where the product of the first two pairwise comparisons is not necessarily equal to the third pairwise comparisons.

A cycle of the above three pairwise comparisons is called a triad. The fundamental challenge for pairwise comparisons (PC matrix for short) is to measure inconsistency in order to decide what is acceptable. Surprisingly, both knowledge and inconsistency can be successfully measured depending on what we wish to achieve although defining them is not easy. For example, we can measure knowledge even by the number of “eighteen wheelers” (large size trucks with 18 wheels used in Northern America for moving cargo) needed to load and move all books when we compare one public library to another. Certainly, such measure is not precise as in one library all books may be identical (e.g., Darwin’s Origin of Species).

The aim of this study is to examine the inconsistency and some inconsistent pairwise comparisons metrics from a geometric interpretation perspective. This

interpretation allows us to deal with infinite matrices and non abelian groups  $G$ . Our approach is based on the works [7], [8] and [9] that we generalize. A summarized presentation of the (classical) pairwise comparisons matrices with coefficients in  $\mathbb{R}_+$  is given in section 2.1, and PC matrices  $i$ -th coefficients in a group  $G$  are described in Section 2.2. The proposed interpretation of PC matrices is described in section 3.

Consider a finite or infinite family  $(s_i)_{i \in I}$ , and the set of 1-vertices  $[s_i, s_j]$ . Consider maps from the set of 1-vertices to a group  $G$ ,

$$[s_i, s_j] \mapsto a_{i,j} \in G.$$

The group  $G$  can be chosen, for example, as  $\mathbb{R}_+^*$  just like in previous works, but one can choose any other group for more complex analysis. Inconsistency is measured by the lack of “morphism of groupoid” property such as

$$a_{i,j} \cdot a_{j,k} \neq a_{i,k}.$$

In order to get more comprehensible measures, the group can be equipped with an indicator map  $ln : G \rightarrow \mathbb{R}_+$ , introduced to measure the lack of consistency, which can be derived from a  $G$ -invariant distance in order to get intuitive pictures. The main motivation for changing  $\mathbb{R}_+^*$  into a more complex group  $G$  is to deal with more complexity, since inconsistency need not be a “linear” notion. Many aspects and measurements can be gathered in one group. Then, depending on the complexity of the problem, one, many or even infinitely many indicator maps can give a more understandable characterization of “acceptable inconsistency”. In [9], by taking some functions  $G \rightarrow \mathbb{R}_+$ , the data on the group  $G$  gives a collection of  $\mathbb{R}_+$ -valued indicator maps. Our family of indexes  $I$  is assumed countable, which means that there exists a one-to-one map  $I \rightarrow \mathbb{Z}$ . Up to re-indexation, we assume  $I = \{0, \dots, n\} \subset \mathbb{N}$ , or  $I = \mathbb{N}$ , or  $I = \mathbb{Z}$ .

Our objectives are:

- to show that the criterion  $a_{i,j} = \lambda_i^{-1} \cdot \lambda_j$  is a necessary and sufficient criterion for consistency, where  $G$  is abelian or not (section 2.2). This criterion was already shown sufficient in the abelian case, in e.g., [8].
- Then, we read the conditions of consistency and inconsistency in the geometric setting of a finite or infinite dimensional simplex (section 3), which can be understood as higher dimensional triangles (which are 2-simplexes) and tetrahedra (which are 3-simplexes), see e.g. [14, 3]. Each  $s_i$  corresponds to a 0-vertex; each 1-vertex gives an edge, and a triad (where inconsistency can be measured) is a 2-face. According to this setting, a geometric picture which is very similar to inconsistency is the holonomy of a connection (see, for example, [10] for holonomy in finite dimensions, and [13] for the infinite dimensional case). We show that a pairwise comparisons matrix  $A$  can always be expressed as a holonomy matrix if the group  $G$  is exponential, and when  $G = (\mathbb{R}_+^*)^J$ , the so-called consistent pairwise comparisons matrices are holonomies of a flat connection which can be constructed.
- Finally, for  $G = \mathbb{R}_+^*$ , to compare pairwise comparisons matrices with what is defined as distance matrices (Section 4), setting (pseudo-)distance along the edges as  $|\ln a_{i,j}|$ , which are generalizations of matrices of curvature in metric spaces [4]. We show that the same distance matrix on a  $n$ -simplex, we can get many comparison matrices, which suggests that the holonomy

picture is more rich in information than the corresponding (pseudo-) distance. However, we also show that for any consistent PC matrix, there is only this consistent PC matrix and its transposition that are consistent and have the same distance matrix.

## 2. INCONSISTENCY IN PAIRWISE COMPARISONS MATRICES

**2.1. Pairwise comparisons matrices with coefficients in  $\mathbb{R}_+^*$ .** It is easy to explain the inconsistency in pairwise comparisons when we consider cycles of three comparisons, called triad and represented here as  $(x, y, z)$ , which do not have the “morphism of groupoid” property such as

$$x.z \neq y$$

Evidently, the inconsistency in a triad  $(x, y, z)$  is somehow (not linearly) proportional to  $y - xz$ . In the linear space (after algorithmic transformation), the inconsistency is measured by the “approximate flatness” of the triangle. The triad is consistent if the triangle is flat. For example,  $(1, 2, 1)$  and  $(10, 101, 10)$  have the difference  $y - xz = 1$  but the inconsistency in the first triad is unacceptable but it is acceptable in the second triad. In order to measure inconsistency, one usually considers coefficients  $a_{i,j}$  with values in an abelian group  $G$ , with at least 3 indexes  $i, j, k$ . The use of “inconsistency” has a meaning of a measure of inconsistency in this study; not the concept itself. The approach to inconsistency (originated in [7] and generalized in [1]) can be reduced to a simple observation:

- search all triads (which generate all 3 by 3 PC submatrices) and locate the worse triad with a so-called inconsistency indicator ( $ii$ ),
- $ii$  of the worse triad becomes  $ii$  of the entire PC matrix.

Expressing it a bit more formally in terms of triads (the upper triangle of a PC submatrix  $3 \times 3$ ), we have:

$$ii(x, y, z) = 1 - \min \left\{ \frac{y}{xz}, \frac{xz}{y} \right\}$$

According to [8], it is equivalent to:

$$ii(x, y, z) = 1 - e^{-|\ln(\frac{y}{xz})|}$$

The expression  $|\ln(\frac{y}{xz})|$  is the distance of the triad  $T$  from 0. When this distance increases, the  $ii(x, y, z)$  also increases. It is important to notice here that this definition allows us to localize the inconsistency in the matrix PC and it is of a considerable importance for most applications.

Another possible definition of the inconsistency indicator can also be defined (following [8]) as:

$$ii(A) = 1 - \min_{1 \leq i < j \leq n} \min \left( \frac{a_{ij}}{a_{i,i+1}a_{i+1,i+2} \dots a_{j-1,j}}, \frac{a_{i,i+1}a_{i+1,i+2} \dots a_{j-1,j}}{a_{ij}} \right)$$

since the matrix  $A$  is consistent if and only if for any  $1 \leq i < j \leq n$  the following equation holds:

$$a_{ij} = a_{i,i+1}a_{i+1,i+2} \dots a_{j-1,j}.$$

It is equivalent to:

$$ii(A) = 1 - \max_{1 \leq i < j \leq n} \left( 1 - e^{-|\ln(\frac{a_{ij}}{a_{i,i+1}a_{i+1,i+2} \dots a_{j-1,j}})|} \right)$$

The first definition of  $ii$  allows us not only find the localization of the worst inconsistency but to reduce the inconsistency by a step-by-step process which is crucial for practical applications. The second definition of  $ii$  is useful when the global inconsistency indicator is needed for acceptance or rejection of the PC matrix. A hybrid of two  $ii$  definitions may be considered in applications.

**2.2. PC matrices with coefficients in a group: extension of known constructions.** Let  $I$  be a set of indexes among  $\mathbb{Z}$ ,  $\mathbb{N}$  or  $\{0, \dots, n\}$  for some  $n \in \mathbb{N}^*$ .

**Definition 2.1.** Let  $(G, .)$  be a group. A **pairwise comparisons matrix** is a matrix

$$A = (a_{i,j})_{(i,j) \in I^2}$$

such that

- (1)  $\forall (i, j) \in I^2, a_{i,j} \in G.$
- (2)  $\forall (i, j) \in I^2, a_{j,i} = a_{i,j}^{-1}.$
- (3)  $a_{i,i} = 1_G.$

**Remark 2.2.** The above definition has been adjusted after receiving remarks from two independent researchers on the earlier version of this text. Condition 3 is a consequence of condition 2 when  $G = \mathbb{R}_+^*$ , but when  $G = S^1 = \{z \in \mathbb{C}, |z| = 1\}$ , condition 3 is necessary. We have to argue that this condition is very natural and intuitive: for self-comparison of an event  $A$ , there is nothing to say more than  $A = A$ , which is traduced by  $a_{i,i} = 1_G$ .

The matrix  $A$  is **consistent** if

$$(2.1) \quad \forall (i, j, k) \in I^3, \quad a_{i,j} \cdot a_{j,k} = a_{i,k}.$$

It implies that:

$$(2.2) \quad \exists (\lambda_i)_{i \in I}, \quad a_{i,j} = \lambda_i^{-1} \cdot \lambda_j \Rightarrow A \text{ is consistent.}$$

**Theorem 2.3.**

$$\exists (\lambda_i)_{i \in I}, \quad a_{i,j} = \lambda_i^{-1} \cdot \lambda_j \Leftrightarrow A \text{ is consistent.}$$

**Proof:** We only need to show the  $\Leftarrow$  part of the proposition by the following two steps: For this,

- (1) we build by induction the family  $(\lambda_i)_{i \in I}$ , in order to get  $a_{i,i+1} = \lambda_i^{-1} \cdot \lambda_{i+1}$ .

Let us fix  $\lambda_0 = 1_G$ . For  $i \geq 0$ , we have the following relation:

$$\lambda_{i+1} = \lambda_i \cdot a_{i,i+1} \quad .$$

If  $I = \mathbb{Z}$ , we add the corresponding relation for  $i \leq 0$  :

$$\lambda_{i-1} = \lambda_i \cdot a_{i-1,i}^{-1} \quad .$$

which gets the desired relation.

- (2) we check that  $\forall (i, k) \in I^2, \quad a_{i,k} = \lambda_i^{-1} \cdot \lambda_k.$

We first notice that  $a_{i,i} = 1_G = \lambda_i^{-1} \cdot \lambda_i$ . Then, for  $i < k$ ,

$$\begin{aligned} a_{i,k} &= \prod_{j=i}^{k-1} a_{j,j+1} = \prod_{j=i}^{k-1} \lambda_j^{-1} \cdot \lambda_{j+1} \\ &= \lambda_i^{-1} \cdot \left( \prod_{j=i+1}^{k-1} \lambda_j \cdot \lambda_j^{-1} \right) \cdot \lambda_k = \lambda_i^{-1} \cdot \lambda_k. \end{aligned}$$

Hence, the property  $a_{j,i} = a_{i,j}^{-1}$  shows what we needed to prove for all the coefficients of the matrix  $A$ . ■

### 3. PC MATRIX READ ON A SIMPLEX

In this study, the group multiplication is used in a contravariant way and written in the reverse order  $(g, g') \mapsto g' \cdot g$ , instead of  $g \cdot g'$ , to reflect that  $G$ -covariance is commonly called as the “right-covariance”. This choice is made for having a more intuitive notation which makes more comprehensive statements when  $G$  is non abelian.

An exposition on holonomy is given in [6, 10, 11, 13]. The geometry of simplexes is well addressed by [3, 14]. Let  $(G, \cdot)$  be a Lie group with Lie algebra  $(\mathfrak{g}, +, [\cdot, \cdot])$ . The expression “Lie group” is here understood in a very general sense. This can be a finite dimensional or an infinite dimensional group, or even a Frölicher group with Lie algebra [13]. The only technical requirement for the sequel is the existence of an exponential map

$$\exp : C^\infty([0; 1], \mathfrak{g}) \rightarrow C^\infty([0; 1], G)$$

solving the logarithmic equation  $g^{-1} \cdot dg = v$ , where  $v \in C^\infty([0; 1], \mathfrak{g})$  and  $g \in C^\infty([0; 1], G)$ . This ensures the existence of the holonomy of a connection [12]. Such a property is always fulfilled for finite dimensional groups, but not for Frölicher Lie groups. We get an example of Frölicher Lie group with no exponential map considering  $G = \text{Diff}_+(]0; 1[)$ , the group of increasing diffeomorphisms of the open unit interval [13]. On a trivial principal bundle  $P = M \times G$ , the horizontal lift of a path  $\gamma \in C^\infty([0; 1], M)$  from a starting point  $p = (\gamma(0), g_0) \in M \times G$  with respect to a connection  $\theta$  is the path  $\tilde{\gamma} = (\gamma, g) \in C^\infty([0, 1], P)$  such that

$$g^{-1} \cdot dg = \theta(d\gamma).$$

If  $\gamma$  is a loop, we have  $\text{Hol}_{g(0)}\gamma = g(0)^{-1} \cdot g(1)$ . The holonomy of a loop depends on the basepoint  $(\gamma(0), g(0))$  and is invariant under coadjoint action. Let  $n \in \mathbb{N}^*$  and

$$\Delta_n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \left( \sum_{i=0}^n x_i = 1 \right) \wedge (\forall i \in \{0, \dots, n\}, x_i \geq 0) \right\}$$

be an  $n$ -simplex. This simplex can be generalized to infinite dimension cases:

$$\Delta_{\mathbb{N}} = \left\{ (x_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N}, \mathbb{R}_+^*) \mid \sum_{i=0}^{\infty} x_i = 1 \right\}$$

and

$$\Delta_{\mathbb{Z}} = \left\{ (x_n)_{n \in \mathbb{Z}} \in l^1(\mathbb{Z}, \mathbb{R}_+^*) \mid \sum_{i \in \mathbb{Z}} x_i = 1 \right\},$$

where the summation over  $\mathbb{Z}$  is done by integration with respect to the counting measure. In the sequel,  $\Delta$  will denote  $\Delta_n$ ,  $\Delta_{\mathbb{N}}$  or  $\Delta_{\mathbb{Z}}$ . Since  $\Delta$  is smoothly contractible, any  $G$ -principal bundle over  $\Delta$  is isomorphic to  $\Delta \times G$  and a  $G$ -connection 1-form on  $\Delta$  is a 1-form  $\theta \in \Omega^1(\Delta, \mathfrak{g})$ , which extends to a  $G$ -covariant 1-form in  $\Omega^1(\Delta, \mathfrak{g})$ , with respect to the coadjoint action of  $G$  on  $\mathfrak{g}$ . We define a gauge  $(\tilde{s}_i)_{i \in I} \in G^I$  with  $\tilde{\gamma}_i(1) = (\gamma_i(1), \tilde{s}_i)$  where

$$\gamma_i = [s_0, s_1] * \dots * [s_{i-1}, s_i] \text{ if } i > 0$$

and

$$\gamma_i = [s_0, s_{-1}] * \dots * [s_{i+1}, s_i] \text{ if } i < 0.$$

Let

$$(3.1) \quad a_{i,j} = s_i^{-1} \cdot \text{Hol}(\gamma_i * [s_i, s_j] * \gamma_j^{-1}) \cdot s_j$$

where the basepoint of holonomy is  $(s_0, 1_G) \in \Delta \times G$ .

Let

$$A = \text{Mat}(a_{i,j}).$$

**Proposition 3.1.** *A is a PC matrix.*

**Proof:** from holonomy in “reverse orientation”, it follows:  $\text{Hol}(\gamma_j * [s_j, s_i] * \gamma_i^{-1}) = \text{Hol}(\gamma_i * [s_i, s_j] * \gamma_j^{-1})^{-1}$ . ■

Let  $\gamma_{i,j,k} = [s_i, s_j] * [s_j, s_k] * [s_k, s_i]$  be the loop based on  $s_i$  along the border of the oriented 2-vertex  $[s_i, s_j, s_k]$ , where  $*$  is the composition of paths. The matrix is consistent if

$$(3.2) \quad \forall i, j, k, \quad a_{i,k} = a_{i,j} \cdot a_{j,k}$$

$$(3.3) \quad \Leftrightarrow a_{i,j} \cdot a_{j,k} \cdot a_{k,i} = a_{i,i} = 1_G$$

$$(3.4) \quad \Leftrightarrow \text{Hol}(\gamma_{i,j,k}) = 1_G$$

By fixing an **indicator map**, defined in [9] as

$$\text{In} : G \rightarrow \mathbb{R}$$

to  $\text{In}(1_G) = 0$ , we get a generalization of the **inconsistency indicator** by setting

$$ii_{\text{In}} = \sup \{ \text{In}(\text{Hol}(\gamma_{i,j,k})) \mid (i, j, k) \in I^3 \}.$$

For example, if  $d$  is a left-invariant distance on  $G$ , a natural indicator map can be:

$$\text{In} : g \mapsto d(1_G, g^{-1}).$$

It needs to be examined whether or not every PC matrix can be expressed as a matrix of holonomies of a fixed connection. For it, we need to assume that the group  $G$  is exponential, with means that the exponential map  $\mathfrak{g} \rightarrow G$  is onto.

**Theorem 3.2.** *If  $G$  is exponential, the map*

$$\begin{aligned} \Omega^1(\Delta, \mathfrak{g}) &\rightarrow \{ \text{PC matrices} \} \\ \theta &\mapsto \text{the holonomy matrix} \end{aligned}$$

*is onto.*

**Proof.** Let  $A = (a_{i,j})_{(i,j) \in I^2}$  be a PC matrix. Let us build a connection 1-form  $\theta \in \Omega^1(\Delta, \mathfrak{g})$  such as (3.1). For this, before constructing our connection, we fix the gauge  $(\tilde{s}_{i \in I}) \in G^I$  by

$$\tilde{s}_i = a_{0,1} \dots a_{i-1,i} \text{ for } i > 0$$

and by

$$\tilde{s}_i = a_{0,-1} \dots a_{i+1,i} \text{ for } i < 0.$$

Once the gauge is fixed, we begin with dealing with each 1-vertex.

Firstly, by fixing indexes  $i < j$ , which holds in particular for  $j = i + 1$ , we choose  $v_{i,j} \in \mathfrak{g}$  such that  $\exp(v_{i,j}) = a_{i,j}$ . Needless to say that the condition  $v_{j,i} = -v_{i,j}$  is consistent with  $a_{i,j}^{-1} = a_{j,i}$ . The group  $\{\exp(tv_{i,j}) | t \in \mathbb{R}\}$  is an abelian subgroup of  $G$ . For this reason, formulas for holonomy on an abelian can be used to specify a function  $f_{i,j} : [0, 1] \rightarrow \mathbb{R}_+ \cdot v_{i,j}$ , with support in  $[1/3; 2/3]$ , and such that  $\int_0^1 f_{i,j}(s) ds = v_i$  on the length-parametrized edge  $[s_i, s_j]$ . Finding such a function is possible, and extending the  $G$ -equivariant 1-form  $f_{i,j} ds$  on  $[s_i, s_j] \times G$  to a  $G$ -equivariant 1-form  $\theta_{i,j}$  on  $\Delta \times G$  which is null off  $V_{i,j} \times G$ , where  $V_{i,j}$  is a tubular neighborhood of radius  $\epsilon > 0$  of  $\text{supp}(f_{i,j})$ , is also possible.

Secondly, we repeat this procedure for each couple of indexes  $(i, j)$  such that  $i < j$ , and choose  $\epsilon$  small enough in order to have non intersecting supports  $\text{supp}(\theta_{i,j})$ , for example  $\epsilon = 1/6$ . By setting

$$\theta = \sum_{i < j} \theta_{i,j},$$

we get a connection  $\theta$  which holonomy matrix is given by  $A$ . ■

Let us provide a geometric criterion for consistency.

**Theorem 3.3.** *If the connection  $\theta$  is flat,  $A$  is consistent.*

**Proof:**  $\theta$  is flat  $\Leftrightarrow$  its curvature is null. This implies that the Lie algebra of the holonomy group is null, and since each 2-vertex  $[s_i, s_j, s_k]$  is contractible, the holonomy group is trivial. ■

Assume that  $G = (\mathbb{R}^*)^J$  where  $J$  is any cardinality, finite or infinite. In this case,

$$\text{Hol}(< s_i, s_j >) = e^{\int_0^1 \theta(ds_{i,j})}$$

where  $ds$  is the unit vector of the normalized length parametrization of  $[s_i, s_j]$ . Thus,

$$(3.5) \quad \text{Hol}(\gamma_{i,j,k}) = 1_G \Leftrightarrow \int_0^1 \theta(ds_{i,j}) + \int_0^1 \theta(ds_{j,k}) + \int_0^1 \theta(ds_{k,i}) = 0$$

The connection  $\theta$  is flat now and it reads as  $d\theta = 0$  which is equivalent to  $\theta = df$ , where  $f \in C^\infty(\Delta, \mathbb{R}^J)$  (because  $H^1(\Delta, \mathbb{R}) = 0$ ). With this function  $f$ , setting  $e^{f(s_i)} = \lambda_i$ , we recover the “basic consistency condition” (2.2). In the spirit of Whitney’s simplicial approximation [14], we assume that  $G = \mathbb{R}^*$  and  $\mathfrak{g} = \mathbb{R}$  for simplicity, and our computations will extend to  $\mathbb{R}^J$  componentwise. Let us construct an affine function  $f$ . This function is uniquely determined by its values  $f(s_i)$ ,

for  $i \in \{0, \dots, n\}$  and we get the system:

$$(3.6) \quad \begin{cases} f(s_0) & -f(s_1) & & = & -\ln(a_{0,1}) \\ & f(s_1) & -f(s_2) & = & -\ln(a_{1,2}) \\ & & \dots & & \\ & & f(s_{n-1}) & -f(s_n) & = & -\ln(a_{n-1,n}) \end{cases}$$

which is a  $n$ -system with  $(n+1)$  variables. Since  $\theta = df$ , we can normalize it, assuming e.g.  $f(s_0) = 0$ , and the system gets a unique solution, and hence a unique affine function  $f$  and an unique connection  $\theta = df$ . Now, setting  $\lambda_i = e^{f(s_i)}$ , we recover the construction given in the proof of 2.3 for this particular choice of group  $G$ .

#### 4. DISTANCE MATRIX

In this section,  $G = \mathbb{R}_+^*$ . Setting

$$k_{i,j} = |\log(a_{i,j})|$$

we get another matrix, that we define as the **distance matrix**

$$K = (k_{i,j})_{(i,j) \in I^2}.$$

Notice that, if the coefficients of this matrix satisfy the triangle inequality  $\forall(i, j, l) \in I^3, k_{i,l} \leq k_{i,j} + k_{j,l}$ , we get a curvature matrix for metric spaces [4]. Due to the absolute value, we have the following:

**Proposition 4.1.** *Let  $K$  be a non zero distance matrix on  $\Delta_n$ . Let  $N$  be the number of non zero coefficients in  $K$ . Then  $N$  is even and there exists  $2^{N/2}$  corresponding PC matrices.*

**Outline of proof.** For each  $k_{i,j} \neq 0$ ,  $\ln(a_{i,j})$  has 2 possible signs. Therefore, we have the following results:

**Theorem 4.2.** *Let  $K$  be the distance matrix on  $\Delta_n$  associated to a consistent pairwise comparisons matrix  $A$ , which is assumed to be non zero. Let  $N'$  be the number of coefficients  $k_{i,i+1}$  which are non zero. Then there exists  $2^{N'}$  consistent PC matrices built with the coefficients  $k_{i,i+1}$ , but only 2 consistent ones,  $A$  and its transposition.*

**Proof:** The first part of the proof follows the last proposition: the sign of  $\ln(a_{i,i+1})$  gives the  $2^{N'}$  consistent pairwise comparisons matrices which correspond to the coefficients  $k_{i,i+1}$ . However, for any coefficient  $k_{i,l}$ , with  $l > i$ , the formula

$$a'_{i,k} = \prod_{j=i}^{l-1} a'_{j,j+1}$$

shows that there are two possible choices:  $\ln a'_{0,1} = \ln a_{0,1}$  or  $\ln a'_{0,1} = -\ln a_{0,1}$ , which determines the sign of the other coefficients of the matrix  $A'$ . ■

**Example:**



Using small values, recommended in [2] for the PC matrix, we use the consistent triad (1.5, 3, 2) to generate the following PC matrix

$$A = \begin{pmatrix} 1 & 1.5 & 3 \\ \frac{2}{3} & 1 & 2 \\ \frac{1}{3} & \frac{1}{2} & 1 \end{pmatrix}$$

with distance matrix:

$$K = \begin{pmatrix} 0 & \ln 1.5 & \ln 3 \\ \ln 1.5 & 0 & \ln 2 \\ \ln 3 & \ln 2 & 0 \end{pmatrix}.$$

The transposed matrix is also coherent:

$$A^t = \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 1.5 & 1 & \frac{1}{2} \\ 3 & 2 & 1 \end{pmatrix}$$

However, the following PC matrix is not coherent:

$$A = \begin{pmatrix} 1 & 2 & \frac{1}{3} \\ \frac{1}{2} & 1 & 2 \\ 3 & \frac{1}{2} & 1 \end{pmatrix}.$$

All these PC matrices have the same distance matrix  $K$ .

## CONCLUSION

Approximate reasoning can be improved by reducing inconsistency in subjective assessments. The matrix  $K$  provides sufficient data to determine the consistency. However,  $K$  may be insufficient to measure inconsistency which requires additional holonomy-like data. This suggests that a global theory of PC matrices only based on the group  $G = \mathbb{R}_+^*$  can be not sufficient to deal with complex situations. Moreover, the theory of PC matrices with coefficients in a non-abelian group demonstrates the same properties as the abelian case, which we hope would be a new direction for future research.

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